Power-Flow Relations in Lossless Nonlinear Media*

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Summary—**The Manley-Rowe relations, originally derived for nonlinear lumped circuit elements, are generalized to include the power flow in the fields produced in the presence of lossless, nonlinear media. The generalization is carried out first for nonlinear anisotropic media with single-valued relations between the instantaneous $\overline{E}$ and $\overline{P}$, and $\overline{H}$ and $\overline{M}$. The proof is extended to include gyromagnetic media under small-signal excitation at the signal frequency (but large excitation at the pump frequency). The relations are applied to show under what conditions power gain can be achieved with a three-frequency and a four-frequency excitation of a ferrite. The form of the coupling coefficients in the electromagnetic operation of a ferrite amplifier is shown to be a consequence of the generalized Manley-Rowe relations.**

I. Introduction

CONSIDERABLE attention has been given recently to amplifiers that employ nonlinear electromagnetic media, are "pumped" at one frequency, and provide gain at one or more frequencies. Good noise performance at microwave frequencies, and other advantages, can be expected from these "parametric" amplifiers, as they are often called. As a specific example of a parametric amplifier, Manley and Rowe[1] considered the nonlinear capacitor, and proved some general relations among the powers flowing into the capacitor terminals at several frequencies. Their results are also applicable to lossless inductors. It is not obvious however, how their power relations can be applied to electromagnetic problems involving nonlinear anisotropic electric and magnetic media. It is also not clear whether their relations are still applicable to nonlinear, nonreciprocal circuits. Devices containing nonlinear gyromagnetic media have equivalent circuits that exhibit nonreciprocal properties. In the construction of the ferromagnetic microwave amplifier one employs just such nonlinear media. Therefore, it seems necessary to extend the Manley-Rowe relations to include the electromagnetic power flow of fields in lossless nonlinear media, and in media which exhibit gyromagnetic effects.

This paper is divided in four parts. First, we derive two expressions for the power flow density (Poynting vector) in nonlinear media at various frequencies. These expressions are used as the basis of the generalization of the Manley-Rowe relations.

Second, we study the relation between the magnetic field intensity $\overline{H}$ and the magnetization $\overline{M}$ for lossless anisotropic magnetic media under the restriction that $\overline{H}$ be a single-valued function of $\overline{M}$.[2] The same is done for the relation between the electric field intensity $\overline{E}$ and the polarization $\overline{P}$ of a lossless anisotropic dielectric material. These relations are then used in the proof of the Manley-Rowe relations for lossless media with single-valued relations between $\overline{H}$ and $\overline{M}$, and $\overline{E}$ and $\overline{P}$.

Third, we apply the well-known relation between $\overline{H}$ and $\overline{M}$ in a gyromagnetic medium to prove the Manley-Rowe relations for lossless gyromagnetic media under small-signal excitation (but, in general, under large excitation at the pump frequency and its harmonics).

Finally, we apply the generalized Manley-Rowe relations to show under what conditions one can achieve gain in a cavity loaded by a ferrite and resonant at the pumping frequency, the signal frequency, and one "idling" frequency. It is found that gain can only be achieved if the idling frequency and signal frequency are both below the pumping frequency. A similar study, carried out for a device operating at the pumping frequency, signal frequency, and two "idling" frequencies, shows that gain can be achieved at frequencies higher than the pump frequency but less than twice the pump frequency. It is shown that an interrelation exists between the coupling coefficients of the equations for the "electromagnetic operation"[3] of a three-frequency (pump, signal, and idling frequency) ferrite amplifier. This interrelation is a consequence of the generalized Manley-Rowe relations.

II. General Power Relations in Nonlinear Media

Suppose that an excitation at two frequencies, ω_0 and ω_1, is applied to a nonlinear medium. The nonlinear medium produces, in general, fields at all frequencies $m\omega_1+n\omega_0$, where m and n are all positive and negative integers. (Components at negative frequencies have the interpretation that is usual in complex Fourier series expansions.) Thus, the electric field at any point $\bar{r}$ can be expanded in the complex Fourier series,

$$\overline{E}(\bar{r}, t) = \sum_{m,n} \overline{E}_{m,n}(\bar{r}) e^{j(m\omega_1+n\omega_0)t}, \tag{1}$$

where the m's and n's run over all integers from $-\infty$ to $+\infty$. Since $\overline{E}(\bar{r}, t)$ is a real vector, we conclude that

$$\overline{E}_{m,n} = (\overline{E}_{-m,-n})^*. \tag{2}$$

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[1] J. M. Manley and H. E. Rowe, "Some general properties of nonlinear elements—Part I. General energy relations," Proc. IRE, vol. 44, pp. 904–914; July, 1956.

[2] We consider $\overline{H}$ to be a single valued function of $\overline{M}$, if $\overline{H}(\bar{r}, t)$ at the point $\bar{r}$ and the time t depends only upon $\overline{M}(\bar{r}, t)$ at the same point and at the same time. $\overline{H}(\bar{r}, t)$ is not supposed to depend upon the time (or space) derivatives of $\overline{M}$. For a linear medium this restriction is equivalent to the requirement that the reciprocity relation be fulfilled.

[3] H. Suhl, "Theory of the ferromagnetic microwave amplifier," *J. Appl. Phys.*, vol. 28, pp. 1225–1236; November, 1957.

Similar expressions apply to the magnetic field $\overline{H}(\bar{r}, t)$, the polarization $\overline{P}(\bar{r}, t)$, and the magnetization $\overline{M}(\bar{r}, t)$. The time-dependent Maxwell equations can be split into Fourier components (one set of equations for each Fourier component). We have

$$\nabla \times \overline{E}_{m,n} = -j\mu_0(m\omega_1 + n\omega_0)(\overline{H}_{m,n} + \overline{M}_{m,n}) \quad (3)$$

$$\nabla \times \overline{H}_{m,n} = j(m\omega_1 + n\omega_0)(\epsilon_0\overline{E}_{m,n} + \overline{P}_{m,n}). \quad (4)$$

Dot-multiplying (3) by

$$\frac{m\overline{H}_{m,n}^*}{m\omega_1 + n\omega_0},$$

the complex conjugate of (4) by

$$\frac{m\overline{E}_{m,n}}{m\omega_1 + n\omega_0},$$

and subtracting the two resulting equations, we obtain, after adding over all m and n,

$$\nabla\cdot \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{m\overline{E}_{m,n} \times \overline{H}_{m,n}^*}{m\omega_1 + n\omega_0} = -j \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} m\overline{M}_{m,n}\cdot\overline{H}_{m,n}^* + j \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} m\overline{P}_{m,n}^*\cdot\overline{E}_{m,n}, \quad (5)$$

where the summations over the products $\overline{E}_{m,n}\cdot\overline{E}_{m,n}^*$ and $\overline{H}_{m,n}\cdot\overline{H}_{m,n}^*$ cancel by virtue of (2) and by an analogous relation for the Fourier components of the magnetic field. Eq. (5) is one of the basic relations for the power flow in nonlinear media which we use in deriving the generalization of the Manley-Rowe relations. Indeed, in order to accomplish the generalization, we only have to prove that the summations on the right-hand side of (5) are zero for the lossless media under consideration.

An equation analogous to (5) can be derived by dot-multiplying (3) by

$$\frac{n\overline{H}_{m,n}^*}{m\omega_1 + n\omega_0},$$

and the complex conjugate of (4) by

$$\frac{n\overline{E}_{m,n}}{m\omega_1 + n\omega_0}.$$

Again, subtracting the two resulting equations, and adding over all m and n, we obtain

$$\nabla\cdot \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{n\overline{E}_{m,n} \times \overline{H}_{m,n}^*}{m\omega_1 + n\omega_0} = -j \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} n\overline{M}_{m,n}\cdot\overline{H}_{m,n}^* + j \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} n\overline{P}_{m,n}^*\cdot\overline{E}_{m,n}. \quad (6)$$

III. Proof of the Manley-Rowe Relations for Lossless "Reciprocal" Media

We call a medium "reciprocal" if a single-valued relation exists between $\overline{H}$ and $\overline{M}$ on the one hand, and $\overline{E}$ and $\overline{P}$ on the other hand. The medium does not have to be isotropic, *i.e.*, $\overline{M}$ does not have to be parallel to $\overline{H}$, and $\overline{E}$ is not necessarily parallel to $\overline{P}$.

Let us study the relation between $\overline{H}$ and $\overline{M}$ that has to hold in order to fulfill the requirement of losslessness. The energy per unit volume supplied to the material in order to produce the magnetization $\overline{M}$ is

$$\int_0^{\overline{M}} \overline{H}\cdot d\overline{M}, \quad (7)$$

where $\overline{H}$ is considered as a function of $\overline{M}$, $\overline{H}(\overline{M})$. Integral (7) has to be single-valued, independent of the "path" in the space of the magnetization vector $\overline{M}$. If $\overline{M}$ is varied, but eventually brought back to its original value, a closed path is described in the space of the magnetization vector $\overline{M}$. The integral over the closed path, $\oint \overline{H}\cdot d\overline{M}$, has to be zero if the medium is lossless and, therefore, no energy is lost in the magnetization. We have

$$\oint \overline{H}\cdot d\overline{M} = 0, \quad (8)$$

for an arbitrary path in the space of the magnetization vector $\overline{M}$. This is possible if, and only if, $\overline{H}$ can be written as the gradient (in the $\overline{M}$ space) of a potential function $U(\overline{M})$:

$$\overline{H} = \nabla_M U(\overline{M}), \quad (9)$$

where

$$\nabla_M = \bar{i}_x \frac{\partial}{\partial M_x} + \bar{i}_y \frac{\partial}{\partial M_y} + \bar{i}_z \frac{\partial}{\partial M_z}.$$

In a similar way, for a lossless anisotropic dielectric medium, we derive

$$\overline{E} = \nabla_P V(\overline{P}), \quad (10)$$

where

$$\nabla_P = \bar{i}_x \frac{\partial}{\partial P_x} + \bar{i}_y \frac{\partial}{\partial P_y} + \bar{i}_z \frac{\partial}{\partial P_z}$$

is the gradient in the space of the polarization vector, $\overline{P}$, in which we may represent $\overline{E}$ as a vector field.

First, let us consider the energy and power relations in a lossless *magnetic* material when two excitations at the angular frequencies ω_0 and ω_1 are applied to the material. The time dependence of the field vectors then contains components at frequencies $m\omega_1+n\omega_0$, where m and n are (positive or negative) integers. At each point in space we must have, for the magnetization vector,

$$\overline{M} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \overline{M}_{m,n}e^{j(m\xi+n\eta)}, \quad (11)$$

where[1]

$$\xi = \omega_1 t \qquad \omega_1 = 2\pi f_1$$

$$\eta = \omega_0 t \qquad \omega_0 = 2\pi f_0$$

$$\overline{M}_{m,n} = \frac{1}{4\pi^2}\int_0^{2\pi} d\eta \int_0^{2\pi} d\xi \overline{M}(\xi, \eta) e^{-j(m\xi+n\eta)}$$

and

$$\overline{M}_{m,n} = (\overline{M}_{-m,-n})^*.$$

$\overline{M}_{m,n}$ is a complex vector function of ξ and η and of the spatial coordinates $\bar{r}=\bar{r}(x, y, z)$. The variables ξ and η can be considered as independent[1] in some mathematical operations. This means that $\overline{M}$ is considered as a function of ξ and η *in the entire* ξ-η *plane*. If we plot each vector component of $\overline{M}(\xi, \eta)$ as the third coordinate in a Cartesian coordinate system (in which ξ, η are the other two coordinates) we obtain three surfaces above the ξ-η plane, one surface for each component of $\overline{M}$. Let us now study the physical significance of these three surfaces. Consider a physical process with particular fundamental excitation frequencies ω_1 and ω_0. Then consider the cuts of the three surfaces $M_x(\xi, \eta)$, $M_y(\xi, \eta)$, and $M_z(\xi, \eta)$ with a plane perpendicular to the ξ-η plane, passing through the origin with the slope $\xi/\eta=\omega_1/\omega_0$. The curves thus obtained are plots of M_x, M_y, and M_z as functions of time for this particular process. A plane with a different slope $\xi/\eta=\omega_1'/\omega_0'$ produces, in general, different curves. These curves are the plots of M_x, M_y, and M_z as functions of time for another physical process, with the fundamental frequencies ω_1' and ω_0'. This new process has the initial values, $M_x(0, 0)$, $M_y(0, 0)$, and $M_z(0, 0)$ at $t=0$, in common with the original process since $\overline{M}(\xi, \eta)$ is a single-valued function of ξ and η. Thus, when we study $\overline{M}(\xi, \eta)$ as a function of ξ, η over the entire ξ-η plane we are studying an infinite number of physical processes with different fundamental frequencies ω_1 and ω_0. All processes have the initial value $\overline{M}(0, 0)$. The time dependence of any particular process (with the frequencies ω_1 and ω_2) is singled out through a cut by a plane perpendicular to the ξ-η plane and intersecting the ξ-η plane along a line through the origin with a slope $\xi/\eta=\omega_1/\omega_0$. For this reason we shall call the line $\xi/\eta=\omega_1/\omega_0$ the "process line." We make use of this picture, in particular, when we are considering gyromagnetic media. Returning now to the problem at hand, we recall that the relation between $\overline{H}$ and $\overline{M}$ is single-valued, by assumption. Hence

$$\overline{H} = \overline{H}[\overline{M}(\xi, \eta)] = \overline{H}(\xi, \eta). \tag{12}$$

$\overline{H}$ can be expanded in a similar Fourier series as $\overline{M}$:

$$\overline{H} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \overline{H}_{m,n} e^{j(m\xi+n\eta)}, \tag{13}$$

where

$$\overline{H}_{m,n} = (\overline{H}_{-m,-n})^* \tag{14}$$

and

$$\overline{H}_{m,n} = \frac{1}{4\pi^2}\int_0^{2\pi} d\eta \int_0^{2\pi} d\xi \overline{H}(\xi, \eta) e^{-j(m\xi+n\eta)}. \tag{15}$$

Dot-multiplying both sides of the complex conjugate of (15) by $jm\overline{M}_{m,n}$, and adding over all m and n, we obtain

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} jm\overline{H}_{m,n}^* \cdot \overline{M}_{m,n} = \frac{1}{4\pi^2}\int_0^{2\pi} d\eta \int_0^{2\pi} d\xi \overline{H}(\xi, \eta) \cdot \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} jm\overline{M}_{m,n} e^{j(m\xi+n\eta)}. \tag{16}$$

The double summation under the integral of (16) can be identified by comparing it with (11):

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} jm\overline{M}_{m,n} e^{j(m\xi+n\eta)} = \frac{\partial \overline{M}}{\partial \xi}.$$

If we introduce this equation into (16), we have

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} jm\overline{H}_{m,n}^* \cdot \overline{M}_{m,n} = \frac{1}{4\pi^2}\int_0^{2\pi} d\eta \int_0^{2\pi} d\xi \overline{H}(\xi, \eta) \cdot \frac{\partial \overline{M}}{\partial \xi}. \tag{17}$$

But $d\xi(\partial\overline{M}/\partial\xi)=d\overline{M}$, with η held constant. Furthermore, according to (9),

$$d\overline{M} \cdot \overline{H} = d\overline{M} \cdot \nabla_M U(\overline{M}) = dU(\overline{M})$$

with η held constant. But $\overline{M}$ is a periodic function of ξ, and U is a single-valued function of $\overline{M}$. Therefore,

$$\int_0^{2\pi} d\xi \frac{d\overline{M}}{d\xi} \cdot \overline{H} = \int_{\overline{M}(0,\eta)}^{\overline{M}(2\pi,\eta)} dU = 0.$$

We thus obtain

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} jm\overline{H}_{m,n}^* \cdot \overline{M}_{m,n} = 0. \tag{18}$$

In a similar way we obtain

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} jn\overline{H}_{m,n}^* \cdot \overline{M}_{m,n} = 0 \tag{19}$$

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} jmP_{m,n}^* \cdot \overline{E}_{m,n} = 0 \tag{20}$$

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} jn\overline{P}_{m,n}^* \cdot \overline{E}_{m,n} = 0. \tag{21}$$

These equations can now be used to derive the extension of the Manley-Rowe relations. Introducing (18) and (20) into (5),we obtain

$$\nabla \cdot \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{m\overline{E}_{m,n} \times H_{m,n}^*}{m\omega_1 + n\omega_0} = 0. \tag{22}$$

In a similar way, when (19) and (21) are introduced into (6) the result is

$$\nabla \cdot \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{n\overline{E}_{m,n} \times \overline{H}_{m,n}^*}{m\omega_1 + n\omega_0} = 0. \tag{23}$$

Eqs. (22) and (23) are the generalizations of the Manley-Rowe relations in differential form.

Let us now imagine that the device under consideration is enclosed by a perfect conductor, except for the openings provided by the feeding guides. Number the various propagating modes in all the guides (each mode in every guide is assigned a particular number) 1, 2, $\cdots$, i, $\cdots$, N. Define, in the usual way, a voltage amplitude $V_{m,n,i}$ and current amplitude $I_{m,n,i}$ for each frequency component $m\omega_1+n\omega_0$ of the ith mode. Now, integrating over the entire volume of the device enclosed by the perfect conductor and by the cross-section planes of the guides, from (22) and (23), we obtain

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{i=1}^{N} \frac{mV_{m,n,i}I_{m,n,i}}{m\omega_1 + n\omega_0} = 0;$$

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{i=1}^{\infty} \frac{nV_{m,n,i}I_{m,n,i}}{m\omega_1 + n\omega_0} = 0.$$

These are the generalizations in equivalent-circuit terminology of the Manley-Rowe relations for microwave devices containing nonlinear anisotropic reciprocal media.

IV. The Small-Signal Manley-Rowe Relations for Gyromagnetic Media

We turn now to the derivation of the Manley-Rowe relations for gyromagnetic media. For this case, only the proof for small-*signal* excitation (but large *pump* excitation) has been found. Since most analyses of gyromagnetic media make use of the small-signal assumptions, publication of the proof at this time seems justified.

The small-signal theory of parametric amplifiers starts with the results of the (in general nonlinear) analysis of the pump excitation in the absence of an applied signal. Let us identify the frequencies ω_1 and ω_0 of Sections II and III with the pump frequency ω_p, and signal frequency ω_s, respectively. The pump excitation produces excitations at the pump frequency ω_p and all its harmonics. A small applied signal produces excitations at the sideband frequencies $m\omega_p+n\omega_s$. Among these, only the sidebands for $n=\pm 1$ are linearly related to the applied signal amplitude. All higher order sidebands, $|n|>2$, contain the applied signal amplitude raised to the nth power and are, therefore, negligible compared to the first order excitation ($|n|=1$) at small applied signal levels.

With this recognition, (6) can be adapted directly to the study of small signal power. Indeed, (6) contains only products of the sideband-excitation amplitudes ($n\neq 0$). If these amplitudes are found from a small-signal analysis to be correct within first order of the applied signal amplitude, the expressions in (6) can be found from them to be correct within second order. All contributions of terms with $|n|>1$ are of higher order than second and can be disregarded in a small-signal analysis. Within the small-signal assumption we thus obtain for (6)

$$\nabla \cdot \sum_{m=-\infty}^{\infty} \sum_{n=\pm 1} \frac{n\overline{E}_{m,n} \times \overline{H}_{m,n}^*}{m\omega_p + n\omega_s} = -j \sum_{m=-\infty}^{\infty} \sum_{n=\pm 1} n\overline{M}_{m,n}\cdot\overline{H}_{m,n}^* + j \sum_{m=-\infty}^{\infty} \sum_{n=\pm 1} n\overline{P}_{m,n}^*\cdot\overline{E}_{m,n}. \tag{24}$$

Eq. (24) is the small-signal counterpart of (6). Any small-signal solution will have to satisfy this equation.

Eq. (5) does not have a small-signal counterpart since it contains cross products of the pumping amplitudes. Small-signal theory disregards the second order changes in these quantities as caused by a small applied signal excitation. However, such second order changes contribute terms of second order to (5). Thus, small-signal theory does not provide the information necessary to use (5) up to second order.

We proceed now to prove the generalized, small-signal Manley-Rowe relations for gyromagnetic media. For this purpose, we must show that the right-hand side of (24) is equal to zero if $\overline{E}$ and $\overline{P}$ on the one hand, and $\overline{H}$ and $\overline{P}$ on the other, fulfill the relations of gyromagnetic media.

We assume that the dielectric characteristics of the material are reciprocal (see Section III for the use of "reciprocal" as applied to nonlinear media). For this case, it has been proved, in general, that

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} jn\overline{P}_{m,n}^*\cdot\overline{E}_{m,n} = 0. \tag{25}$$

For a small applied signal, all terms in (25) with $|n|\neq 1$ are negligible, and thus we have

$$\sum_{m=-\infty}^{\infty} \sum_{n=\pm 1} jn\overline{P}_{m,n}^*\cdot\overline{E}_{m,n} = 0. \tag{26}$$

Having proved that the second term on the right-hand side of (24) is zero, we turn our attention to the first term. We must prove that

$$\sum_{m=-\infty}^{\infty} \sum_{n=\pm 1} jn\overline{M}_{m,n}\cdot\overline{H}_{m,n}^* = 0, \tag{27}$$

subject to the condition that $\overline{M}$ and $\overline{H}$ must satisfy the equation of a gyromagnetic medium. We have

$$\dot{\overline{M}} = -\gamma(\overline{M} \times \overline{H}), \tag{28}$$

where $\overline{M}$ and $\overline{H}$ are the time-dependent magnetization and magnetic field, containing both the large-signal and small-signal parts.

First, consider the magnetization $\overline{M}(t)$. It consists of the large-signal part $\overline{M}_0(t)$ produced by the pumping excitation. Aside from a time-average component, this part has only Fourier components at ω_p and its harmonics. The small perturbation $\overline{M}_1(t)$ of $\overline{M}(t)$, produced by a small signal at the frequency ω_s, has Fourier components at $m\omega_p \pm \omega_s$.

Thus

$$\overline{M}(t) = \overline{M}_0(t) + \overline{M}_1(t), \tag{29}$$

where

$$\overline{M}_0(t) = \sum_{m=-\infty}^{\infty} \overline{M}_m e^{jm\omega_p t} \tag{30}$$

$$\overline{M}_1(t) = \sum_{m=-\infty}^{\infty} \sum_{n=\pm 1} \overline{M}_{m,n} e^{j(m\omega_p + n\omega_s)t}. \tag{31}$$

With an analogous separation of the magnetic field into a small-signal part and a large-signal part, we obtain for (28)

$$\dot{\overline{M}}_1 = -\gamma(\overline{M}_0 \times \overline{H}_1 + \overline{M}_1 \times \overline{H}_0). \tag{32}$$

Eq. (32) determines $\overline{H}_1$ in terms of $\overline{M}_1$. Note, however, that it does not yield the component of $\overline{H}_1$ parallel to $\overline{M}_0$, $\overline{H}_1^{\parallel}$, since that component cancels when it is cross-multiplied by $\overline{M}_0$. Thus, (32) gives a relation only for $\overline{H}_1^{\perp}$, the component of $\overline{H}_1$ perpendicular to $\overline{M}_0$. The component $\overline{H}_1^{\parallel}$ is entirely independent of $\overline{M}_1$. Let us introduce, again, the variables

$$\xi = \omega_p t; \qquad \eta = \omega_s t. \tag{33}$$

In terms of these variables we may rewrite $\overline{M}_1$ formally as a function of ξ and η:

$$\overline{M}_1 = \overline{M}_1(\xi, \eta) = \sum_{m=-\infty}^{\infty} \sum_{n=\pm 1} \overline{M}_{m,n} e^{j(m\xi + n\eta)}. \tag{34}$$

The time derivative of $\overline{M}_1$ for any particular choice of ω_p and ω_s, *i.e.*, for any particular "process-line" in the ξ-η plane, is

$$\dot{\overline{M}}_1 = \omega_p \frac{\partial \overline{M}_1}{\partial \xi} + \omega_s \frac{\partial \overline{M}_1}{\partial \eta} \cdot \tag{35}$$

We may now introduce (35) into (32), in order to obtain $\overline{H}_1$ as a function of ξ and η. According to (30) and (33),

$$\overline{M}_0 = \sum_{n=-\infty}^{\infty} \overline{M}_m e^{jm\xi} = \overline{M}_0(\xi). \tag{36}$$

Cross-multiplying (32) by $M_0(\xi)$ and expressing H_0 as a function of ξ analogously to (36) we have

$$\overline{H}_1^{\perp}(\xi, \eta) = \frac{1}{\gamma} \frac{1}{M_0^2} \left[\overline{M}_0(\xi) \times \left[\omega_p \frac{\partial \overline{M}_1(\xi, \eta)}{\partial \xi} + \omega_s \frac{\partial \overline{M}_1(\xi, \eta)}{\partial \eta} \right] + \gamma \overline{M}_0(\xi) \times [\overline{M}_1(\xi, \eta) \times \overline{H}_0(\xi)] \right]. \tag{37}$$

Eq. (37) gives the component of the small-signal H field perpendicular to $\overline{M}_0$. Note that $\overline{M}_0^2$ is a constant, independent of ξ. Indeed, from (28), applied to the pumping excitation alone, we have

$$\dot{\overline{M}}_0 = -\gamma(\overline{M}_0 \times \overline{H}_0)$$

or, if we use (36) and a corresponding equation for $\overline{H}_0$, we obtain

$$\omega_p \frac{\partial \overline{M}_0(\xi)}{\partial \xi} = -\gamma(\overline{M}_0(\xi) \times \overline{H}_0(\xi)). \tag{38}$$

Dot-multiplying (38) by $\overline{M}_0$, we have

$$\omega_p \frac{\partial \overline{M}_0(\xi)}{\partial \xi} \cdot \overline{M}_0(\xi) = \frac{1}{2} \omega_p \frac{\partial}{\partial \xi} (\overline{M}_0^2(\xi)) = 0.$$

Thus, $\overline{M}_0^2$ does not depend upon ξ.

The component of $\overline{H}_1$ parallel to $\overline{M}_0$, $\overline{H}_1^{\parallel}$, is independent of $\overline{M}_1$. It can obviously be written as

$$\overline{H}_1^{\parallel}(\xi, \eta) = \overline{M}_0(\xi) f(\xi, \eta) \tag{39}$$

where $f(\xi, \eta)$ is a periodic scalar function of ξ and η.

Having expressed all time functions as function of ξ and η, we are ready now to construct the proof for (27) analogously to the derivation of (17). We have

$$\sum_{m=-\infty}^{\infty} \sum_{n=\pm 1} jn \overline{M}_{m,n} \cdot \overline{H}_{m,n}^* = \frac{1}{4\pi^2} \int_0^{2\pi} d\xi \int_0^{2\pi} d\eta \overline{H}(\xi, \eta) \sum_{m=-\infty}^{\infty} \sum_{n=\pm 1} jn \overline{M}_{m,n} e^{j(m\xi + n\eta)} = \frac{1}{4\pi^2} \int_0^{2\pi} d\xi \int_0^{2\pi} d\eta \frac{\partial \overline{M}_1}{\partial \eta} \cdot \overline{H}_1. \tag{40}$$

If we split $\overline{H}_1(\xi, \eta)$ into its components parallel and perpendicular to $\overline{M}_0$, we can write for the integral on the right-hand side of (40)

$$\frac{1}{4\pi^2} \int_0^{2\pi} d\xi \int_0^{2\pi} d\eta \frac{\partial \overline{M}_1}{\partial \eta} \cdot \overline{H}_1 = \frac{1}{4\pi^2} \int_0^{2\pi} d\xi \int_0^{2\pi} d\eta \frac{\partial \overline{M}_1}{\partial \eta} \cdot \overline{H}_1^{\perp} + \frac{1}{4\pi^2} \int_0^{2\pi} d\xi \int_0^{2\pi} d\eta \frac{\partial \overline{M}_1}{\partial \eta} \cdot \overline{H}_1^{\parallel}. \tag{41}$$

In the appendix, using (37) and (39), the proof is presented that the integrals on the right-hand side of (41) equal zero. This completes the generalization of the Manley-Rowe relations to gyromagnetic media under small-signal excitation. From (24), for these we have

$$\nabla \cdot \sum_{m=-\infty}^{\infty} \sum_{n=\pm 1} \frac{n \overline{E}_{m,n} \times \overline{H}_{m,n}^*}{m\omega_p + n\omega_s} = 0. \tag{42}$$

The integral form of (42) is obtained by interating it over a volume enclosed by the surface S

$$\sum_{m=-\infty}^{\infty} \sum_{n=\pm 1} \oint \frac{n \overline{E}_{m,n} \times \overline{H}_{m,n}^*}{m\omega_p + n\omega_s} \cdot d\overline{S} = 0. \tag{43}$$

V. Application of the Manley-Rowe Relations

The Manley-Rowe relations indicate what devices can be realized either by using nonlinear reciprocal or gyromagnetic media. Among the devices that are of interest is the parametric amplifier. A particular version of this is a device containing a nonlinear medium to which power is *supplied*[4] at the pump frequency. A signal is applied to it at a frequency ω_s. Other frequencies are not produced directly but only through the nonlinear action of the medium.

Let us study first a particularly simple, but important, case of a parametric amplifier in which the only frequency components with a finite power flow are those of the pump frequency, of the small signal, and of one sideband, the "idling" frequency (either at the frequency $\omega_p+\omega_s$, or at the frequency $\omega_p-\omega_s$). This can be accomplished if the nonlinear medium is inside a cavity that is resonant at ω_p, ω_s and $\omega_p \pm \omega_s$, but not at any other of the frequencies $m\omega_p+n\omega_s$. The small-signal Manley-Rowe relation (43) then reduces to

$$\text{Re} \oint \left[\frac{\overline{E}_{0,1} \times \overline{H}_{0,1}^*}{\omega_s} \pm \frac{\overline{E}_{1,\pm 1} \times \overline{H}_{1,\pm 1}^*}{\omega_p \pm \omega_s} \right] \cdot d\overline{S} = 0, \quad (44)$$

in which the integration is carried over a surface enclosing the nonlinear medium. The surface vector $d\overline{S}$ points outward from the surface. Note that the power flow at the pump frequency does not appear explicitly in the small-signal expression (44).

We are interested in obtaining power gain at the signal frequency with no other power than the pump power *supplied* to the medium. The relation

$$\text{Re} \oint \overline{E}_{1,\pm 1} \times \overline{H}_{1,\pm 1}^* \cdot d\overline{S} \geqq 0 \quad (45)$$

indicates that, at the (idling) frequency $\omega_p \pm \omega_s$, we *extract* power from, rather than supply it to, the medium, Using inequality (45) in (44), we obtain

$$\mp \left[\frac{\omega_p \pm \omega_s}{\omega_s} \right] \text{Re} \oint \overline{E}_{0,1} \times \overline{H}_{0,1}^* \cdot d\overline{S} \geqq 0.$$

Thus, we get power *out* of the medium at the desired frequency, ω_s, provided that 1) $\omega_s < \omega_p$ and 2) the side band at which power is extracted from the medium is a lower sideband at frequency $\omega_p - \omega_s$ (see Fig. 1). However, if $\omega_s > \omega_p$ and/or the other sideband that is used is at $\omega_p + \omega_s$, no power can be extracted from the medium. This proves the following theorem.

Theorem

A small-signal parametric amplifier that uses a "reciprocal" or gyromagnetic medium cannot have power gain at a frequency higher than the pump frequency, if a finite power flow is associated only with three frequencies ω_s, ω_p, $\omega_p \pm \omega_s$. However, this theorem does not entirely exclude the possibility of gain at a signal frequency higher than the pump frequency. Indeed, if we allow for finite amplitudes at *four* frequencies, ω_p, ω_s, $\omega_p+\omega_s$, and $\omega_p-\omega_s$,

[4] In this sense the signal source does not supply any power to the medium, since more power flows out of the medium at the signal frequency than flows into it, if gain is to be obtained.

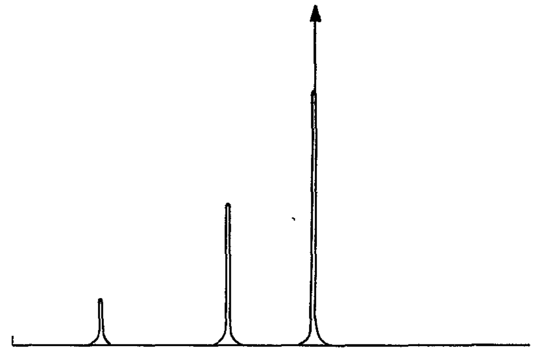

Fig. 1—Spectrum for three-frequency parametric device with possibility of gain.

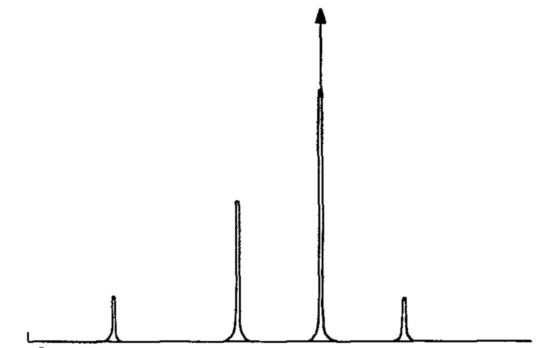

Fig. 2—Spectrum for four-frequency parametric device with gain.

then from (43), we obtain

$$\text{Re} \left[\oint \frac{\overline{E}_{0,1} \times \overline{H}_{0,1}^*}{\omega_s} \cdot d\overline{S} + \oint \frac{\overline{E}_{1,1} \times \overline{H}_{1,1}^*}{\omega_p + \omega_s} \cdot d\overline{S} - \oint \frac{\overline{E}_{1,-1} \times \overline{H}_{1,-1}^*}{\omega_p - \omega_s} \cdot d\overline{S} \right] = 0. \quad (46)$$

Now, if we assume that $\omega_s < \omega_p$, but with $\omega_p+\omega_s$ as the signal frequency, and ω_s and $\omega_p-\omega_s$ as the idling frequencies, we find that extraction of power from the medium at frequency $\omega_p-\omega_s$ permits positive values of the integral Re $\oint \overline{E} \times \overline{H}^* \cdot d\overline{S}$ at both frequencies ω_s and $\omega_p+\omega_s > \omega_p$. Thus, the Manley-Rowe relations do not prohibit power gain at a frequency $\omega_p+\omega_s$. The spectrum of the four frequencies that are employed is shown in Fig. 2.

If, however, $\omega_s > \omega_p$, then not all integrals Re $\oint \overline{E} \times \overline{H}^* \cdot d\overline{S}$ in (46) can be positive. Thus, power gain is impossible at signal frequencies higher than twice the pump frequency. We have thus proved the following theorem.

Theorem

The generalized Manley-Rowe relations permit power gain in a parametric amplifier at a frequency $\omega_p+\omega_s$ in the range $\omega_p < \omega_p+\omega_s < 2\omega_p$, if a finite power flow is associated with four frequencies, ω_s, $\omega_p-\omega_s$, $\omega_p+\omega_s$, and ω_p.

Now we turn to another application of the generalized Manley-Rowe relations. Eq. (43) imposes restrictions upon the general form of the coupling equations for cavity modes in the presence of ferrites. The limitations on the length of this paper do not permit a detailed derivation. A brief summary of the results obtained should suffice. For this purpose we are going to consider Suhl's analysis[3] as an example. Eqs. (13) and (14) of Suhl[3] give the equations of (weak) coupling produced by a ferrite sample between the amplitudes A_1 and A_2 of the resonant modes of a cavity at the frequencies ω_1 and ω_2, where $\omega_1+\omega_2=\omega_p$. These equations are

$$\left(2\lambda + \frac{\omega_1}{Q_1}\right) A_1 = \rho_{12} A_2^*$$

$$\left(2\lambda + \frac{\omega_2}{Q_2}\right) A_2 = \rho_{21} A_1^*$$

where λ gives the (slow) time variation of the modes as $\exp(\lambda t)$. It can be shown that (43) imposes on the coupling coefficients ρ_{12} and ρ_{21} the restriction

$$\frac{\rho_{12}}{\omega_1}\int_{\text{cavity}} h_1^2 dv = \frac{\rho_{21}}{\omega_2}\int_{\text{cavity}} h_2^2 dv \tag{47}$$

where h_1 and h_2 are the normalized fields of the modes 1 and 2. Detailed calculations of the coupling coefficients, as done by Suhl,[3] confirm (47).

VI. Conclusions

It has been found that the electromagnetic Poynting vectors (pertaining to various frequencies) of fields in "reciprocal" media obey (22) and (23). When equivalent circuit terminology is introduced into the field problem, relations result that are very similar in appearance to the Manley-Rowe relations.

A small-signal form of one of the generalized Manley-Rowe relations, (43), has been proved for gyromagnetic media. The relation can be used to determine under what circumstances power gain can be expected from a gyromagnetic medium under parametric excitation. The same relation can also be used to predict the form of the equations for the coupling between cavity modes produced by a "parametrically" excited ferrite. It is worth mentioning that (43) can be used as the basis of a theory of mode coupling produced by a parametrically excited uniform ferrite rod in a uniform microwave structure. This application of (43) is reminiscent of the use of the kinetic power theorem[5] as the basis for the traveling-wave tube analysis.[6]

VII. Appendix

We shall prove that

$$\int_0^{2\pi} d\xi \int_0^{2\pi} d\eta \frac{\partial \overline{M}_1}{\partial \eta}\cdot \overline{H}_1^{\parallel} = 0 \tag{48}$$

and

$$\int_0^{2\pi} d\xi \int_0^{2\pi} d\eta \frac{\partial \overline{M}_1}{\partial \eta}\cdot H_1^{\perp} = 0. \tag{49}$$

Consider, first, the integrand in (49) and use (37). Then we have

$$\frac{\partial \overline{M}_1}{\partial \eta}\cdot H_1^{\perp} = \frac{1}{\gamma}\frac{1}{\overline{M}_0^2}\left\{\omega_p \overline{M}_0(\xi) \times \frac{\partial \overline{M}_1(\xi,\eta)}{\partial \xi}\cdot\frac{\partial \overline{M}_1(\xi,\eta)}{\partial \eta} + \gamma \overline{M}_0(\xi)\cdot[\overline{M}_1(\xi,\eta)\times \overline{H}_0(\xi)]\times\frac{\partial \overline{M}_1(\xi,\eta)}{\partial \eta}\right\}. \tag{50}$$

A glance at (34) shows that $\overline{M}_1(\xi,\eta)$ can be split into

$$\overline{M}_1(\xi,\eta) = \overline{m}_+(\xi)e^{j\eta} + \overline{m}_-(\xi)e^{-j\eta}, \tag{51}$$

where

$$\overline{m}_+(\xi) = \sum_{m=-\infty}^{\infty} \overline{M}_{m,+1}e^{jm\xi}$$

and

$$\overline{m}_-(\xi) = \sum_{m=-\infty}^{\infty} \overline{M}_{m,-1}e^{jm\xi}.$$

For the sake of brevity, we omit, henceforth, the explicit indication of arguments ξ and η. Let us consider the integral over ξ and η of the first term in (50).

$$I_1 = \int_0^{2\pi} d\xi \int_0^{2\pi} d\eta \frac{1}{\gamma}\frac{1}{\overline{M}_0^2}\left(\omega_p \overline{M}_0 \times \frac{\partial \overline{M}_1}{\partial \xi}\cdot\frac{\partial \overline{M}_1}{\partial \eta}\right). \tag{52}$$

If we introduce (51) into (52) for $\overline{M}_1$, we note that only products of $\overline{m}_+$ and $\overline{m}_-$ will remain after integration of η over the periodic interval of η.

$$I_1 = \int_0^{2\pi} d\xi \frac{2\pi j}{\gamma \overline{M}_0^2}\omega_p \overline{M}_0\cdot\left(\frac{\partial m_-}{\partial \xi}\times \overline{m}_+ - \frac{\partial \overline{m}_+}{\partial \xi}\times \overline{m}_-\right)$$
$$= \int_0^{2\pi} d\xi \frac{2\pi j}{\gamma \overline{M}_0^2}\omega_p \overline{M}_0\cdot\frac{\partial}{\partial \xi}(\overline{m}_- \times \overline{m}_+). \tag{53}$$

Next, let us look at the integration over η of the second term in (50). We note that

$$\frac{1}{\gamma \overline{M}_0^2}\overline{M}_0\cdot(\overline{M}_1\times\overline{H}_0)\times\frac{\partial \overline{M}_1}{\partial \eta}$$
$$= \frac{1}{\gamma \overline{M}_0^2}\left[\left(\overline{M}_1\cdot\frac{\partial \overline{M}_1}{\partial \eta}\right)\overline{M}_0\cdot\overline{H}_0 - \left(\overline{H}_0\cdot\frac{\partial \overline{M}_1}{\partial \eta}\right)\overline{M}_1\cdot\overline{M}_0\right]. \tag{54}$$

Integrating this term over one period of η, and using a simple vector identity, we obtain

$$I_2 = \int_0^{2\pi} d\xi \frac{1}{\gamma \overline{M}_0^2}\int_0^{2\pi} d\eta\left[\overline{M}_0\cdot(\overline{M}_1\times\overline{H}_0)\times\frac{\partial \overline{M}_1}{\partial \eta}\right]$$
$$= \int_0^{2\pi} d\xi \frac{1}{\gamma \overline{M}_0^2}\left[(\overline{M}_0\cdot\overline{H}_0)\int_0^{2\pi}\frac{\partial}{\partial \eta}\left(\frac{1}{2}\overline{M}_1^2\right)d\eta + 2\pi j \overline{H}_0\times(\overline{m}_+\times\overline{m}_-)\cdot\overline{M}_0\right]. \tag{55}$$

[5] L. J. Chu, "A kinetic power theorem," presented at 1951 IRE Conf. on Electron Devices, Durham, N. H.; June, 1951.

[6] J. R. Pierce, "Coupling of modes of propagation," *J. Appl. Phys.*, vol. 24, pp. 179–183; February, 1954.

The first term on the right-hand side of (55) equals zero. Combining (53) and (55), for the integral of (50) we obtain

$$I_1 + I_2 = \int_0^{2\pi} d\xi \int_0^{2\pi} d\eta \, \frac{\partial \overline{M}_1}{\partial \eta} \cdot \overline{H}_1{}^{\perp}$$

$$= \frac{2\pi j}{\gamma \overline{M}_0{}^2} \int_0^{2\pi} d\xi \left[\omega_p \overline{M}_0 \cdot \frac{\partial}{\partial \xi} (\overline{m}_- \times \overline{m}_+) - \gamma(\overline{M}_0 \times \overline{H}_0) \cdot (\overline{m}_- \times \overline{m}_+) \right]. \quad (56)$$

However, if we use (38), we find that the integrand of (56) reduces to a total differential

$$\omega_p \overline{M}_0 \cdot \frac{\partial}{\partial \xi} (\overline{m}_- \times \overline{m}_+) - \gamma(\overline{M}_0 \times \overline{H}_0) \cdot (\overline{m}_- \times \overline{m}_+) = \omega_p \frac{\partial}{\partial \xi} (\overline{M}_0 \cdot \overline{m}_- \times \overline{m}_+).$$

The integral over one period in ξ of a total derivative with respect to ξ is zero. Thus, we have proved the correctness of (49). Finally, we have to prove (48). First, we note that $\overline{H}_1{}^{\parallel}$ is parallel to $\overline{M}_0$. Thus it is possible to write $\overline{H}_1{}^{\parallel}$ in the form

$$\overline{H}_1{}^{\parallel} = \overline{M}_0(\xi) f(\xi, \eta), \quad (39)$$

where $f(\xi, \eta)$ is an arbitrary scalar function of ξ and η. Thus,

$$\frac{\partial}{\partial \eta} (\overline{M}_1) \cdot \overline{H}_1{}^{\parallel} = \frac{\partial}{\partial \eta} (\overline{M}_1 \cdot \overline{M}_0) f(\xi, \eta). \quad (57)$$

However, from (32) and (28), we have

$$\dot{\overline{M}}_1 \cdot \overline{M}_0 + \overline{M}_1 \cdot \dot{\overline{M}}_0 = -\gamma(\overline{M}_1 \times \overline{H}_0 + \overline{M}_0 \times \overline{H}_1) \cdot \overline{M}_0 - \gamma(\overline{M}_0 \times \overline{H}_0) \cdot \overline{M}_1 = 0. \quad (58)$$

Thus, for every "process-line" originating at the source of the ξ-η plane, we have

$$\omega_p \frac{\partial}{\partial \xi} (\overline{M}_1 \cdot \overline{M}_0) + \omega_s \frac{\partial}{\partial \eta} (\overline{M}_1 \cdot \overline{M}_0) = 0. \quad (59)$$

Therefore $\overline{M}_1 \cdot \overline{M}_0$ is constant along every process line. At the origin, all process lines have the same value of $\overline{M}_1 \cdot \overline{M}_0$. Hence, $\overline{M}_1 \cdot \overline{M}_0$ is constant throughout the entire ξ-η plane and

$$\frac{\partial (\overline{M}_1 \cdot \overline{M}_0)}{\partial \eta} = 0.$$

Accordingly, (57) equals zero. This proves the correctness of (48).

VIII. ACKNOWLEDGMENT

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[7] M. T. Weiss, "Quantum derivation of energy relations analogous to those for nonlinear reactances," PROC. IRE, vol. 45, pp. 1012–1013; July, 1957.

One Aspect of Minimum Noise Figure Microwave Mixer Design*

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Summary—**A theory is derived which enables a direct measurement of the optimum RF impedance for minimum noise figure. This is achieved by an extension of Pound's method for loss measurements. Also, an analysis is made of the relation between minimum noise figure and maximum gain of the mixer represented as a two-port network.**

The procedure consists of first matching the RF signal input terminals with short-circuited IF terminals. Next open-circuited IF terminal conditions are obtained by a circuit used by Pound. Then a reference plane is determined coinciding by preference with the plane of a maximum in the standing wave pattern of VSWR $= r$. A discontinuity is finally introduced that would have a VSWR of $\rho = \sqrt{r}$ and have its maximum or minimum at the plane of reference.

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INTRODUCTION

MICROWAVE mixer performance has been treated in the literature [1]–[3]. In this paper the mixer is represented by a two-port network. It is assumed that the network has been optimized on an image-frequency termination basis. The aspect treated